This paper discusses several theoretical issues related to the score function for the measurement-to-track association/assignment decision in the track-oriented version of the multiple hypothesis tracker (MHT). This score function is the likelihood ratio: the ratio of the probability density function (pdf) of a measurement having originated from a track, to the pdf of this measurement having a different origin. The likelihood ratio score is derived rigorously starting from the fully Bayesian MHT (hypothesis oriented, based on combinatorial analysis of the general multitarget problem), which is shown to be amenable under some (reasonable) assumptions to the track-oriented MHT (TOMHT). The latter can be implemented efficiently using multidimensional assignment (MDA). The main feature of a likelihood ratio is the fact that it is a (physically) dimensionless quantity and, consequently, can be used for the association of different numbers of measurements and/or measurements of different dimension. The explicit forms of the likelihood ratio are discussed both for the commonly used Kalman tracking filter, as well as for the interacting multiple model (IMM) estimator. The issues of measurements of different dimension and different coordinate systems together with the selection of certain MHT design parameters—the spatial densities of the false measurements and new targets—are also discussed.

I. INTRODUCTION

The multiple hypothesis tracking (MHT) technique originated in [13] as a data association and target tracking technique that evaluates the probabilities of sequences of measurements having originated from various targets. While expensive to implement at the time, current computing power and cost make it feasible for many problems. It has been discussed in several texts, e.g., [4, 8, 1, 5], and has gained significant popularity. The MHT technique is typically implemented using likelihood-based scores of the various individual tracks—this is the “track-oriented” MHT (TOMHT), proposed in [8]. This is a “bottom-up” approach, which starts with feasible measurement to track association hypotheses and builds global (joint) hypotheses subject to “one-to-one” feasibility constraints [10].

The original, fully Bayesian, “hypothesis-oriented” MHT (HOMHT) was developed to yield direct probabilities of global (joint) measurement to target association hypotheses; however, its implementation is substantially more complex than that of the TOMHT. The original implementation of the HOMHT [13, 1] was by enumeration of the feasible global hypotheses and calculation of their probabilities, from which individual track probabilities could be obtained by marginalization, i.e., a “top-down” approach. The downside of the TOMHT is that, unlike the HOMHT, it does not provide global hypothesis probabilities. Nevertheless, since the TOMHT is simpler and conveniently implementable via multidimensional assignment (MDA), using Lagrangian relaxation [10, 6, 11, 12], it is the preferred one.

While the scoring of the various association hypotheses in the TOMHT was presented in [4] as likelihood ratios,1 their derivation following the pioneering work of [14] and the original full-fledged multitarget Bayesian approach [13] did not appear in the literature. The purpose of the present paper is to remedy this and also to present the correct methods of hypothesis scoring for the situations of missing measurements or measurements of different dimensions as they follow from [13]. MHT association probabilities and likelihood functions are also discussed in [16]. Likelihood ratios have been used for sequential testing of tracks in [4], [5], and [17]. The handling of unresolved measurements in MHT was discussed in [7] and [16].

Section II describes the procedure to obtain the likelihood ratio of an individual track, so that it is clearly dimensionless, from the fully Bayesian MHT (hypothesis-oriented, based on combinatorial analysis of the general multitarget

1Unfortunately, no explicit scoring equations were given in [8], which led to several flawed implementations.
II. HYPOTHESIS-ORIENTED AND TRACK-ORIENTED MHT APPROACHES

The HOMHT technique [13] evaluates the probabilities of various sequences of measurements having originated from various targets in a multitarget environment. This is accomplished by setting up a Bayesian framework, which consists of a target motion model, target measurement model, and models for false measurements and new target birth. The approach is to set up, at every sampling time, all the possible hypotheses as continuations/branchings of the hypotheses at the previous sampling time.

As shown in [1, eq. (6.3.3-18)], the HOMHT yields the following probability of a cumulative joint event (global hypothesis up to time $k$)

$$P\{\Theta_{k}^{\text{ij}} | Z_{k}\} = \frac{1}{c \cdot m(k)!}\cdot P_{d}(\phi)P_{s}(\nu)\cdot P_{f}(1/P_{d}(k))^{\phi}\cdot P_{f}(1/P_{d}(k))^{\nu}\cdot \prod_{j=1}^{m(k)}\left\{ f_{j}(\phi_{j}(k) | \Theta_{k}^{\text{ij}1}, Z_{k}^{1:k})\right\}^{\nu_{j}}$$

where the following notations are used:

- $[\Theta_{k}^{\text{ij}}]$ joint association hypothesis $l$ through the current time $k$,
- $[Z_{k}^{1:k}]$ cumulative set of observations through the current time $k$,
- $[\phi]$ the number of measurements (at the current time) deemed as false alarms in the hypothesis under consideration,
- $[\nu]$ the number of measurements (at the current time) deemed as new targets in the hypothesis under consideration,
- $[m(k)]$ the number of measurements at time $k$—this should be interpreted as a scan/frame/list,

2The derivation presented here is different from the one in the preliminary version of this paper [3].


\[
\frac{1}{c^{m}}(\lambda_{0})^{\delta}(\lambda_{c})^{\gamma} \prod_{j=1}^{m} \left(f_{j}(z_{j}(k))\right)^{\gamma} \\
\cdot \prod_{i} \left[ P_{D_{i}}(k) \right]^{\delta} \left[ 1 - P_{D_{i}}(k) \right]^{\gamma} = \left\{ \Theta^{k-1} | Z^{k-1} \right\}.
\]

(4)

Note that the fraction following the normalization constant \(c\) above is the same for all events \(\Theta^{k,j}\) and thus it can be incorporated into the new constant \(c'\).

The likelihood function (the “evidence from the data”) of a variable/event is defined (in a static problem) as the pdf/probability of the observation(s) conditioned on the variable/event of interest [2]. Equivalently, this is the term in Bayes’ formula (not including the normalization factor) that multiplies the prior to obtain the posterior of the variable/event of interest.

In the present (dynamic) problem, the likelihood function for measurement \(z_{j}(k)\) having originated from track \(t\) is the term (without the normalization factor) multiplying the parent hypothesis probability. Consequently, one has from (4) the “continuation” likelihood as

\[
\Lambda_{tj}(k) = f_{j}(z_{j}(k))\, P_{D_{t}}(k).
\]

If measurement \(z_{j}(k)\) has a “false detection” origin, the multiplying term it contributes to (4) is

\[
\Lambda_{tj}(k) = \gamma_{t},
\]

while if its origin is a new target, then

\[
\Lambda_{tj}(k) = \lambda_{t},
\]

Consequently, since both of these situations correspond to associating this measurement to the “dummy track” (indexed as 0 [10]), one has

\[
\Lambda_{00}(k) = \Lambda_{t0}(k) + \Lambda_{tj}(k) = \gamma_{t} + \lambda_{t} \quad \Delta \lambda_{ex}
\]

(8)

where \(\lambda_{ex}\) is the total spatial density of the “extraneous” measurements.

Consequently, the continuation likelihood ratio—that measurement \(z_{j}(k)\) originated from track \(t\) versus being extraneous—is

\[
\mathcal{L}_{tj}(k) \triangleq \frac{\Lambda_{tj}(k)}{\Lambda_{00}(k)} = \frac{f_{j}(z_{j}(k)) P_{D_{t}}(k)}{\lambda_{ex}}.
\]

If no measurement is associated with track \(t\) at time \(k\) (i.e., this track is associated with the dummy measurement [10], indexed as 0), the “no-continuation” likelihood function is

\[
\Lambda_{00}(k) = 1 - P_{D_{t}}(k).
\]

Therefore the no-continuation likelihood ratio is

\[
\mathcal{L}_{00}(k) \triangleq \frac{\Lambda_{00}(k)}{\Lambda_{00}(k)} = 1 - P_{D_{t}}(k).
\]

The above spatial densities (from the corresponding Poisson processes) can be interpreted as uniform distributions in the “average volume occupied by one such measurement” (the inverse of the spatial density). It should be noted that in deriving (1), the extraneous measurements are modeled with pdf \(V^{-1}\). However, through the derivations based on the Poisson model, the spatial density \(\lambda_{ex}\) ends up playing the role of this pdf. The spatial density \(\lambda\) can be interpreted, loosely, as being the pdf of the “nearest” extraneous measurement. The above are important observations in obtaining these quantities in practice since they are algorithm design parameters.

It is easy to see that (9) is (physically) dimensionless: the pdf in its numerator has its physical dimension the inverse of the (physical) dimension of the measurement, i.e., the same as the denominator. Note that (12) is also a dimensionless quantity. The discussion from [16] presents association probabilities but not dimensionless likelihood ratios.

It is absolutely essential to use the (physically dimensionless) likelihood ratios to be able to compare hypotheses consisting of different numbers of measurement associations—the corresponding likelihood functions have different physical dimensions and cannot be numerically compared, just like an area and a volume. This has been overlooked in certain implementations.

The likelihood ratio \(\mathcal{L}_{tj}(k)\) in (9) is detailed later (under the additional Gaussian assumption) in (20).

III. IMPLEMENTATION OF MHT

The original HOMHT [13] was implemented by constructing the tree with all the feasible joint hypotheses—as a top down approach, from joint hypotheses to tracks. This was then followed by an enumerative/exhaustive evaluation of their probabilities according to (1). Gating [1] was obviously used to eliminate low probability associations and low probability hypotheses were eliminated while similar hypotheses (with common recent history—in a sliding window) were combined. Nevertheless, this enumerative approach can quickly become very time consuming.

The TOMHT [8] is a bottom up approach—starting with tracks, joint hypotheses are formed. This was originally also implemented in the same
enumerate, leading to similar limitations. Consequently, efficient coding was vital to make it practical [5]. More recently, the use of MDA\(^3\) algorithms [10, 11] allowed a systematic approach to the search through the tree of the joint hypotheses. This is discussed in more detail next.

### A. Implementation of TOMHT

The “two-way” origin of a measurement (target versus extraneous), shown in (9) is the basis for the track-oriented MHT. Typically, \(\lambda_\phi\) is a subjective quantity and the Poisson model for the appearance of new targets is only a mathematical idealization. This indicates that one can use \(\lambda_\phi\) (which has a physical basis in false alarms and clutter) augmented by what is deemed a “reasonable” \(\lambda_\phi\). The reason this works is that any measurement deemed false is always used later in the track-oriented MHT as an initiator for a new track.

Another advantage of this two-way measurement origin approach is that it lends itself easily to the use of optimization-based search algorithms (MDA) to find the most likely set of tracks. In a sliding window MHT implementation\(^4\) one has \(N\) lists, the oldest of which are “frozen” (firmed up) tracks, followed by the most recent \(N - 1\) scans/frames of measurements. Each item from each list should be assigned once and only once into an \(N\)-tuple consisting of exactly one item from each of the \(N\) lists. Each list also has to have a “dummy” element [10], which has no constraints on how many times it can be assigned. The dummy element in the track list stands for “false track,” while in the measurement lists it stands for “no measurement.” This makes the assignment into a “complete” one, with exactly one item from each of the \(N\) lists.

The common algorithms for obtaining the most likely hypothesis used are the modified auction or Jonker-Volgenant-Castanon (JVC) for 2-D assignment (\(N = 2\), i.e., minimal time depth of \(N - 1 = 1\)) and the same combined with Lagrangian relaxation [10] for the multidimensional case (\(N > 2\), for which problem is NP-hard; the Lagrangian relaxation provides a near-optimal solution). In order to obtain the top \(M\) most likely joint hypotheses, one has to use the \(M\)-best MDA [12].

### B. Implementation of HOMHT

While it is easier to implement the TOMHT (with its two-way measurement origin assumption) using discrete optimization, one can also implement in a similar manner the HOMHT with its three-way measurement origin. In this case the list of tracks shall have two kinds of dummy elements [9] (with no restrictions on their use): false track and new track. Searching for the best \(M\) joint (global) hypotheses with the algorithm of [12] can make this approach systematic and efficient by avoiding the enumerative procedure. The key point is obtaining a separable cost function, which can be accomplished as follows. Since

\[
\phi + \nu + \sum_{j=1}^{m(k)} \tau_j = m(k)
\]

one can divide the r.h.s. of (4) by

\[
(\lambda_\phi)^{m(k)} = (\lambda_\phi)^{\phi + \nu + \sum_{j=1}^{m(k)} \tau_j}
\]

and absorb \((\lambda_\phi)^{-m(k)}\) into \(c\) to preserve the equality. This leads, after cancelation of \((\lambda_\phi)^0\), to

\[
P\{\Theta^k \mid Z^k\} = \frac{1}{c} \left(\frac{\lambda_\phi}{\lambda_\phi}\right)^{w(k)} \prod_{j=1}^{m(k)} \left\{ \frac{f_j(z_j(k))}{\lambda_\phi} \right\}^\gamma 
\cdot \prod_t \left\{ [P_D(k)]^\delta (1 - P_D(k))^{1-\delta} \right\} P\{\Theta^{k-1,t} \mid Z^{k-1} \}.
\]

From the above the likelihood ratio for track continuation is

\[
L_{ij} = \frac{f_j[z_j(k)]}{\lambda_\phi} P_D(k).
\]

The likelihood ratio for no continuation is

\[
L_{i0} = 1 - P_D(k)
\]

and for new target (versus false) the likelihood ratio is

\[
L_{i\phi} = \frac{\lambda_\phi}{\lambda_\phi}.
\]

Since they multiply the parent track’s probability, one has simple additions to be performed for the log-likelihood ratio (LLR), as shown later in (19). Furthermore, these expressions are independent across tracks; this is the key separability requirement for the discrete optimization, where the assignment cost (total score, with opposite sign, so instead of score maximization one has a cost minimization problem) for a set of tuples is always the sum of the costs (negative scores) for the individual tuples.

Thus the TOMHT is the same as the HOMHT with \(\lambda_\phi = 0\) and then, as discussed earlier, each false measurement is treated as a potential new target.

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\(^1\)Also referred to as S-D (S-dimensional) assignment or multiframe assignment (MFA).

\(^4\)This is the only practical way to implement the MHT because otherwise its requirements increase exponentially with time.
IV. LOG-LIKELIHOOD RATIO OF A TRACK

The usual criterion to find the best assignment is by minimizing a cost function, which in this case is the negative of the score function (the likelihood ratio) presented above. For numerical convenience, one uses the natural logarithm, which leads to the negative log-likelihood ratio (NLLR), to be denoted by $\ell$. The cumulative NLLR of track $t$ through time $k$ (the score function) is the sum

$$
\ell_t^k \triangleq -\ln \mathcal{L}_t^k = -\sum_{l=0}^{k} \ln \mathcal{L}_{t, j(t, l)}(l) = \sum_{l=0}^{k} \ell_{t, j(l, l)}(l)
$$

(19)

where the likelihood ratio of associating target $t$ at time $l$ to measurement $j = j(t, l)$ is given in (9) for $j > 0$ and by (12) for $j = 0$. For simplicity we use the subscript $j$ without its arguments in the sequel. Equation (19) is a consequence of the fact that the innovations in a track are white [2], i.e., their joint pdf is the product of the marginal pdfs and thus their logs are added up for the cumulative LLR.

A. KALMAN FILTER AS TRACKER

Assuming the tracker to be a Kalman filter with Gaussian innovations, the NLLR at time $k$ corresponding to (9) is given by

$$
\ell_{j}(k) \triangleq -\ln \mathcal{L}_{j}(k) = \frac{1}{2}[z_{j}(k) - \hat{z}_{j}(k | k - 1)]S_{j}(k)^{-1}[z_{j}(k) - \hat{z}_{j}(k | k - 1)]
$$

$$
+ \ln \frac{\lambda_{\text{ex}}}{P_{D_{j}}(k)} + \ln \frac{2\pi S_{j}(k)}{P_{D_{j}}(k)}^{1/2}
$$

(20)

if measurement $z_{j}(k)$ is assigned to track $t$, which has the predicted measurement $\hat{z}_{j}(k | k - 1)$ with covariance $S_{j}(k)$ (the innovation or residual covariance, which, in general can depend on the candidate measurement as well, because different measurements can have different accuracies). The NLLR corresponding to (12) is

$$
\ell_{0}(k) \triangleq -\ln \mathcal{L}_{0}(k) = -\ln[1 - P_{D_{0}}(k)]
$$

(21)

if there is no measurement assigned (i.e., has the dummy measurement, indexed by 0). In the above, $P_{D_{j}}(k)$ is the detection probability of track $j$ at time $k$ and $\lambda_{\text{ex}}$ is the spatial density of the extraneous measurements (expected number per unit volume in frame/scan $k$). Note that the numerator of the last term in (20) is proportional (but not equal) to the expected number of the extraneous measurements in the validation region of track $j$ at time $k$.

As shown in the previous section, this follows exactly from [1, eq. (6.3-18)], which is based on the original (hypothesis-oriented) MHT [13], after using a single Poisson pmf for the number of the extraneous measurements (which combine the false measurements and the new targets). The same result was also obtained in [5, eq. (6.19)] but in a somewhat different manner, following the original work of [14], as extended in [15].

REMARKS Note that the lack of $\lambda_{\text{ex}}$ in (20) would not affect the result of comparing $\mathcal{L}_{j(t, j)}$ with $\mathcal{L}_{j(0)}$ (i.e., whether measurement $j_1$ or $j_2$ should be associated with track $t$) because $\lambda_{\text{ex}}$, which acts as a common scaling of the likelihood ratios, cancels in the comparison. This is the case with eq. (6.29) of [5]. However, the result of comparing $\mathcal{L}_{j(t, j)}$ with $\mathcal{L}_{j(0)}$ (i.e., whether measurement $j_1$ or no measurement should be associated with track $t$) would be affected by the measurement units used if $\lambda_{\text{ex}}$ is missing. The use of $P_{D_{j}}$ in place of $\lambda_{\text{ex}}$, as in some implementations, does not solve this physical dimensionality problem because the former is dimensionless.

Since the cumulative NLLR (19) can be used in a Wald sequential test (see [5]) to confirm or delete a tentative track, the choice of $\lambda_{\text{ex}}$ does have an impact on this test. Thus $\lambda_{\text{ex}}$ is an algorithm design parameter and it should be selected reflecting as accurately as possible the sensors and the environment. The connection between $\lambda_{\text{ex}}$ and different sensors and the environment is discussed further in Section IVD.

B. IMM Estimator as Tracker

In this case the likelihood function of model $n$ of target (track) $t$ with measurement $z_{j}(k)$ is (see [2, eq. (11.6.6-11)])

$$
\Lambda_{nt}(k) = \mathcal{N}(z_{j}(k); \hat{z}_{nt}(k | k - 1), S_{ntj}(k))
$$

(22)

where $\hat{z}_{nt}(k | k - 1)$ and $S_{ntj}(k)$ are the predicted measurement and its (possibly measurement dependent) covariance for model $n$ of the IMM tracker of target $t$. These (mode-conditioned) innovation pdfs are approximated as Gaussian.

Following the mixture assumption of the IMM estimator (see [2, sec. 11.6.6]) the overall likelihood function of associating $z_{j}(k)$ with target $t$ is the following mixture (weighted sum)

$$
\Lambda_{tj}(k) = \sum_{n=1}^{r} \Lambda_{ntj}(k) \mu_{nt}(k | k - 1)
$$

(23)

where $r$ is the number of models (modes) in the IMM estimator and $\mu_{nt}(k | k - 1)$ is the predicted probability for model $n$ to be in effect at time $k$ from the previous time $k - 1$, which is given by

$$
\mu_{nt}(k | k - 1) = \sum_{l=1}^{r} p_{ln} \mu_{lt}(k - 1)
$$

(24)

where $p_{ln}$ is the Markov chain jump probability from model $l$ (at $k - 1$) to model $n$ (at $k$) and $\mu_{lt}(k - 1)$ is
the updated probability of model $l$ for target $t$ at $k - 1$ (see [2, eq. (11.6.6-14)]).

Based on the above, the track continuation NLLR (9) for the IMM estimator is

$$
\ell_{ij}(k) \equiv -\ln L_{ij}(k) = -\ln \frac{\Lambda_{ij}(k)P_{D_i}(k)}{\lambda_{\text{ex}}}
$$

where $\Lambda_{ij}(k)$ is given in (23). The no-continuation NLLR is the same as in (21).

C. Initialization of Likelihood Ratio

From the above discussion it follows that the initialization of (20) for $k = 1$ should be done with (18). For $k = 2$, expression (20) of the likelihood ratio can be used directly with a suitable definition of the predicted measurement. Assuming the state consists of position and velocity, this can be done as follows.

1) Define the estimated position components of the state at $k = 1$ as the position measurement at $k = 1$, with its covariance matrix as the position measurement noise covariance matrix.

2) The velocity estimate at $k = 1$ can be taken as zero with standard deviation in each component equal to, say, half the maximum speed, uncorrelated between components and uncorrelated with the position.

3) With the above defined state estimate at $k = 1$, use the standard expression for the predicted measurement $\hat{z}(2 \mid 1)$.

D. Handling of Measurements of Different Dimension and Different Coordinate Systems

For measurements of different dimension, the same (dimensionless) expression of the likelihood ratio as in (20) should be used except that the spatial density of the extraneous measurements is defined in the appropriate space.

For example, if the usual measurement (assuming a 3-D radar) is range ($r$), azimuth ($\alpha$), elevation ($\varepsilon$), and range rate ($\dot{r}$), it can happen that there is no range rate in some of the measurements.

Denote the resolution cell in the 4-D $r$-$\alpha$-$e$-$\dot{r}$ sensor spherical-Doppler space as $V_{\text{rae}}$. Then if the false alarm probability in such a resolution cell is $P_{F_{\alpha\text{rae}}}$, the spatial density of the false measurements in this space is

$$
\lambda_{\alpha\text{rae}} = \frac{P_{F_{\alpha\text{rae}}}}{V_{\text{rae}}}
$$

where

$$
V_{\text{rae}} = C_rC_\alpha C_\varepsilon C_\dot{r}
$$

is the 4-D resolution cell volume.

Let the measurement space (surveillance region) volume be

$$
V_{\text{rae}} = (r_{\text{max}} - r_{\text{min}})(\alpha_{\text{max}} - \alpha_{\text{min}})(\varepsilon_{\text{max}} - \varepsilon_{\text{min}})(\dot{r}_{\text{max}} - \dot{r}_{\text{min}})
$$

where $\dot{r}_{\text{min}}$ is, typically, the negative of $\dot{r}_{\text{max}}$. The above assumes the surveillance region to be, in position, a spherical sector.

With $n_e$ being the expected number of new targets in a frame/scan, their spatial density is

$$
\lambda_{\text{exrae}} = \frac{n_e}{V_{\text{rae}}}
$$

assuming homogeneity in the spherical-Doppler coordinates. Clearly, this is a subjective quantity based on mathematics, which should be combined with engineering/physical common sense. Homogeneity of the new targets is more plausible in Cartesian coordinates; this is discussed at the end of this subsection, where the new target spatial density in $r$-$\alpha$-$e$-$\dot{r}$ is presented assuming homogeneity in Cartesian coordinates.

Equations (26) and (29) yield the extraneous target density according to (8) as

$$
\lambda_{\text{exrae}} = \lambda_{\alpha\text{rae}} + \lambda_{\text{exrae}}.
$$

Then the pdf $f_{(r_{\alpha e})}$ of a measurement in the $r$-$\alpha$-$e$-$\dot{r}$ space, when divided by (30), yields a dimensionless likelihood ratio, which can be used in (19).

If a 3-D measurement, consisting of range ($r$), azimuth ($\alpha$), and elevation ($\varepsilon$), is obtained, then one has a false alarm density

$$
\lambda_{\alpha\text{rae}} = \frac{P_{F_{\alpha\text{rae}}}}{V_{\text{rae}}}
$$

where

$$
V_{\text{rae}} = C_rC_\alpha C_\varepsilon
$$

is the 3-D resolution cell volume. Depending on the signal processing, the false alarm probabilities in (26) and (31) might be the same or different.

Similarly, the new target spatial density in a frame/scan in this case, assuming homogeneity in the spherical coordinates, given by

$$
\lambda_{\text{exrae}} = \frac{n_e}{V_{\text{rae}}}
$$

where the volume of the 3-D surveillance space is, assuming it to be a spherical sector

$$
V_{\text{rae}} = (r_{\text{max}} - r_{\text{min}})(\alpha_{\text{max}} - \alpha_{\text{min}})(\varepsilon_{\text{max}} - \varepsilon_{\text{min}})
$$

and $n_e$ is the expected number of new targets in a frame/scan. Since, as discussed above, homogeneity of the new targets is, however, more plausible in Cartesian coordinates, the new target spatial density in $r$-$\alpha$-$e$ assuming homogeneity in Cartesian coordinates is presented at the end of this subsection.

Again, the total spatial density of extraneous measurements follows from (8) with (31) and (33) and the likelihood ratio for this case is obtained by dividing the pdf $f_{(r_{\alpha e})}$ of a measurement in the $r$-$\alpha$-$e$ space with this spatial density, yielding a dimensionless quantity.

Situation of Converted Measurements: If the spherical measurements are converted into Cartesian
position\(^5\) \(x\)-\(y\)-\(z\), then the spatial density of the false and new targets measurements has to be calculated in this space. The volume of the resolution cell (which is defined in the sensor spherical coordinates) becomes in Cartesian coordinates

\[
V_{xyz} = C_r r \cos e C_{\alpha} \rho C_{\varepsilon}
\]  

(35)
i.e., it is location dependent (the term \(r \cos e\) is the horizontal radius, used for the azimuth cell). This is then to be used in the calculation of the spatial density of the false alarms (which will be also location dependent; one can use the measured range and elevation in its evaluation) as follows

\[
\lambda_{\phi_{xyz}} = \frac{P_{FA_{xyz}}}{V_{xyz}}
\]  

(36)
where \(P_{FA_{xyz}}\) is the same as in (31) but it is divided by a 3-D Cartesian volume.

The new target spatial density, assuming homogeneity in the Cartesian space, is

\[
\lambda_{\nu_{xyz}} = \frac{n_f}{V_{xyz}}
\]  

(37)
where the Cartesian volume of the 3-D surveillance space is, assuming it to be a spherical sector

\[
V_{xyz} = \frac{1}{3} (r_3^3 - r_1^3) (a_3 - a_1) (\sin e_3 - \sin e_1).
\]  

(38)

Then using (8) with (36) and (37), yield the spatial density of the extraneous measurements in Cartesian coordinates

\[
\lambda_{\phi_{xyz}} = \lambda_{\phi_{xyz}} + \lambda_{\nu_{xyz}}.
\]  

(39)

The pdf \(f_{xyz}\) of a measurement in the 3-D Cartesian \(x\)-\(y\)-\(z\) space, when divided by this spatial density, yields a dimensionless likelihood ratio.

**New Target Density in Radar Coordinates Assuming Homogeneity in Cartesian Coordinates:** Making the more plausible assumption that the new targets have a spatial density that is constant in Cartesian coordinates, yields the new targets density in \(r\)-\(\alpha\)-\(e\) as

\[
\lambda_{\nu_{xyz}} = \lambda_{\nu_{xyz}} \frac{\Delta V_{xyz}(\Delta r, \Delta a, \Delta e)}{\Delta V_{rad}(\Delta r, \Delta a, \Delta e)}
\]  

(40)
where the infinitesimal Cartesian volume corresponding to increments \(\Delta r, \Delta a, \Delta e\) is, from (38), given by

\[
\Delta V_{xyz}(\Delta r, \Delta a, \Delta e) = \frac{1}{2} [(\rho + \Delta \rho)^3 - \rho^3] \Delta a [\sin(e + \Delta e) - \sin e] = r^2 \cos e \Delta r \Delta a \Delta e.
\]  

(41)

Since

\[
\Delta V_{xyz}(\Delta r, \Delta a, \Delta e) = \Delta r \Delta a \Delta e
\]  

(42)

one obtains from (40)

\[
\lambda_{\nu_{xyz}} = \lambda_{\nu_{xyz}} r^2 \cos e.
\]  

(43)
A similar argument yields

\[
\lambda_{\nu_{xyz}} = \lambda_{\nu_{xyz}} \frac{r^2 \cos e}{r_{\max} - r_{\min}}
\]  

(44)
asuming uniform distribution of the range rate.

V. SUMMARY AND CONCLUSION

This paper discussed in detail the likelihood ratio score function for measurement-to-track assignment with emphasis on the fact that it is a (physically) dimensionless quantity. This guarantees that units do not affect the results and allows seamless handling of missing measurements and measurements of different (vector) dimension—these are very important practical considerations. The derivation of the track-oriented MHT from the original hypothesis-oriented MHT has been presented. The implementation of these two algorithms with MDA has been described. The specific association score functions when using a Kalman filter and an IMM tracker have been presented. Also the issues of measurements of different dimension and different coordinate systems have been discussed.

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Yaakov Bar-Shalom (S’63—M’66—SM’80—F’84) was born on May 11, 1941. He received the B.S. and M.S. degrees from the Technion, Israel Institute of Technology, in 1963 and 1967 and the Ph.D. degree from Princeton University, Princeton, NJ, in 1970, all in electrical engineering.

From 1970 to 1976 he was with Systems Control, Inc., Palo Alto, CA. Currently he is Board of Trustees Distinguished Professor in the Dept. of Electrical and Computer Engineering and Marianne E. Klewin Professor in Engineering. He is also director of the ESP Lab (Estimation and Signal Processing) at the University of Connecticut. His research interests are in estimation theory and stochastic adaptive control and he has published over 350 papers and book chapters in these areas. In view of the causality principle between the given name of a person (in this case, “(he) will track,” in the modern version of the original language of the Bible) and the profession of this person, his interests have focused on tracking.


During 1976 and 1977 he served as associate editor of the IEEE Transactions on Automatic Control and from 1978 to 1981 as associate editor of Automatica. He was program chairman of the 1982 American Control Conference, general chairman of the 1985 ACC, and cochairman of the 1989 IEEE International Conference on Control and Applications. During 1983–1987 he served as chairman of the Conference Activities Board of the IEEE Control Systems Society and during 1987–1989 was a member of the Board of Governors of the IEEE CSS. Currently he is a member of the Board of Directors of the International Society of Information Fusion and served as its Y2K and Y2K2 President. In 1987 he received the IEEE CSS distinguished Member Award. Since 1995 he is a distinguished lecturer of the IEEE AESS. He is coreipient of the M. Barry Carlton Awards for the best paper in the IEEE Transactions on Aerospace and Electronic Systems in 1995 and 2000, and received the 1998 University of Connecticut AAUP Excellence Award for Research and the 2002 J. Mignona Data Fusion Award from the DoD JDL Data Fusion Group.
Samuel S. Blackman received the B.A. degree in mathematics from the University of Hawaii, Honolulu, in 1960 and the M.S. degree in physics from the University of New Mexico, Albuquerque in 1963.

He has been employed by Hughes Aircraft Company (now Raytheon) since 1963 and has worked principally in the development of tracking systems. His current research interests include the applications of tracking and estimation techniques to ground targets.

Mr. Blackman is the author of *Multiple Target Tracking with Radar Applications*, Artech House (1986), *Design and Analysis of Modern Tracking Systems*, Artech House (1999), and numerous papers related to estimation and tracking. He is a member of Phi Beta Kappa.

Robert J. Fitzgerald was born in Canada and received a bachelor’s degree from the University of Toronto and a Ph.D. from the Massachusetts Institute of Technology (both in mechanical engineering). He also studied at the École Nationale Superieure de l’Aéronautique in Paris.

Since 1964 he has been with various divisions of the Raytheon Company, where his activities have included the application of estimation, control, and error analysis techniques to target tracking, interceptor guidance, aided inertial navigation, ballistic missile defense, and radar analysis. Applications have included guidance of missiles, torpedoes, and spin-stabilized projectiles; missile-borne radar (Patriot, Sparrow, AMRAAM); ground and shipboard phased arrays (Cobra Dane, Cobra Judy, PAVE PAWS, Patriot, THAAD, GBR-P, XBR, UEWR, DD-X Dual-Band Radar); dish radars (TARTAR, HAWK, SPS-49, ASDE, Have Stare); and over-the-horizon radar (ROTHR).

He is currently an Engineering Fellow in Raytheon’s Integrated Defense Systems segment, where he contributes technically to a variety of radar projects, principally in the development and evaluation of software algorithms for target tracking and various other aspects of radar operation.