In this article, a numerical technique is presented for the solution of Fokker–Planck equation. This method uses the cubic B-spline scaling functions. The method consists of expanding the required approximate solution as the elements of cubic B-spline scaling function. Using the operational matrix of derivative, the problem will be reduced to a set of algebraic equations. Some numerical examples are included to demonstrate the validity and applicability of the technique. The method is easy to implement and produces very accurate results. © 2008 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 25: 418–429, 2009

Keywords: Fokker-Planck equation; cubic B-spline; wavelets; scaling functions; nonlinear dynamics

I. INTRODUCTION

Fokker-Planck equation arises in a number of different fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology, and circuit theory. A Fokker-Planck equation describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The Fokker–Planck equation was first used by Fokker and Planck (for instance, see [1]) to describe the Brownian motion of particles. If a small particle of mass $m$ is immersed in a fluid, the equation of motion for the distribution function $W(v,t)$ is given by:

$$\frac{\partial W}{\partial t} = \gamma \frac{\partial v W}{\partial v} + \gamma \frac{K T}{m} \frac{\partial^2 W}{\partial v^2},$$

(1.1)

where $v$ is the velocity for the Brownian motion of a small particle, $t$ is the time, $\gamma$ is the fraction constant, $K$ is Boltzmann’s constant, and $T$ is the temperature of fluid [1]. Equation (1.1) is one of the simplest type of Fokker–Planck equations. By solving Eq. (1.1) starting with distribution

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function $W(v, t)$ for $t = 0$ and subject to the appropriate boundary conditions, one can obtain [2] the distribution function $W(v, t)$ for $t > 0$.

The general Fokker—Planck equation for the motion of a concentration field $u(x, t)$ of one space variable $x$ at time $t$ has the form [1–5]

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \quad (1.2)$$

with the initial condition

$$u(x, 0) = f(x), \quad x \in \mathbb{R}, \quad (1.3)$$

where $u(x, t)$ is unknown, $B(x) > 0$ is the diffusion coefficient, and $A(x) > 0$ is the drift coefficient. The drift and diffusion coefficients may also depend on time i.e.

$$\frac{\partial u}{\partial t} = \left[ -\frac{\partial}{\partial x} A(x, t) + \frac{\partial^2}{\partial x^2} B(x, t) \right] u. \quad (1.4)$$

Equation (1.1) is seen to be a special case of the Fokker—Planck equation where the drift coefficient is linear and the diffusion coefficient is constant. Equation (1.2) is an equation of motion for the distribution function $u(x, t)$. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to $x$. In the mathematical literature, Eq. (1.2) is also called forward Kolmogorov equation [2]. The similar partial differential equation is a backward Kolmogorov equation that is [1] in the form:

$$\frac{\partial u}{\partial t} = \left[ -A(x, t) \frac{\partial}{\partial x} + B(x, t) \frac{\partial^2}{\partial x^2} \right] u. \quad (1.5)$$

A generalization of equation (1.2) to $N$ variables $x_1, \ldots, x_N$ has the form:

$$\frac{\partial u}{\partial t} = \left[ -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(X) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(X) \right] u, \quad (1.6)$$

with the initial condition

$$u(X, 0) = f(X), \quad X \in \mathbb{R}^N, \quad (1.7)$$

where $X = (x_1, \ldots, x_N)$. The drift vector $A_i$ and diffusion tensor $B_{i,j}$ generally depend on the $N$ variables $x_1, \ldots, x_N$.

One may find analytical solutions of the Fokker—Planck equation. Generally, however, it is difficult to obtain solutions, especially if no separation of variables is possible or if the number of variables is large.

Various methods of solution are: simulation methods, transformation of a Fokker—Planck equation to a Schrödinger equation, numerical integration methods, etc. [1].

There is a more general form of Fokker—Planck equation. Nonlinear Fokker—Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics,
laser physics, pattern formation, psychology, and marketing (see [4] and references therein). In one variable case the nonlinear Fokker—Planck equation is written in the following form:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x, t, u) + \frac{\partial^2}{\partial x^2} B(x, t, u)\right] u. \quad (1.8)$$

For $N$ variables $x_1, \ldots, x_N$, it has the form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(X, t, u) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(X, t, u)\right] u, \quad (1.9)$$

where $X = (x_1, \ldots, x_N)$. Notice that when $A_i(X, t, u) = A_i(X)$ and $B_{i,j}(X, t, u) = B_{i,j}(X)$ the nonlinear Fokker—Planck equation (1.9) reduces to the linear Fokker-Planck equation (1.6).

It is worth noting that some semi-analytic techniques are employed to solve the Fokker-Planck equation. For example this equation is investigated in [2] using the Adomian decomposition method. Also the variational iteration method is developed in [4] to solve this equation. However, in this work we propose an alternative approach. For some other investigations on this model or some other similar models the interested readers can see references [6–10]. Authors of [11] developed a finite difference technique [12–15] to solve the type of Fokker-Planck equations describing the stochastic dynamics of a particle in a storage ring. One important problem in accelerator physics is to study the dynamics of charged particles under the influence of electromagnetic fields and noise. These models lead to stochastic differential equations in six-dimensional phase space or equivalently to the Fokker-Planck equation [11]. In [16] a finite difference procedure is given for solving the Fokker-Planck equation in two dimensions. The method is tested with problems where analytical solutions exist and it is compared with a finite element scheme. For more applications of the model studied in this work the interested reader can see [5, 17, 18].

It is worth pointing out that in this work we propose an alternative approach. In the current investigation, we reduce the problem to a set of algebraic equations by expanding the unknown function as cubic B-spline scaling functions specially constructed on bounded interval, with unknown coefficients. The operational matrix of derivative is given. This matrix together with the cubic B-spline scaling functions are then utilized to evaluate the unknown coefficients.

This article is organized as follows: In Section II, we describe the formulation of the cubic B-spline scaling functions on $[0, 1]$, and construct the dual functions and then derive the operational matrices of derivative and integral required for our subsequent development. In Section III, the proposed method is used to approximate the solution of the problem in interval $[0, 1]$ for variables $x$ and $t$. As a result, a set of algebraic equations are formed and a solution of the considered problem is introduced. In Section IV, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting several numerical examples. Section V, ends this article with a brief conclusion. Note that we have computed the numerical results by Maple programming.

Employing the scheme developed in this report to solve some partial differential equations studied in [19–21] and [24–28] can be useful investigation.

II. CUBIC B-SPLINE SCALING FUNCTIONS ON $[0, 1]$

Scaling functions can be used to expand any functions in $L^2(\mathbb{R})$. These functions are defined on the entire real lines, so that they could be outside of the domain of the problem [23]. To avoid this
occurrence, compactly supported spline scaling functions, constructed for the bounded interval
[0, 1], have been taken into account in this article.

Let $m$ and $n$ be positive integers and

$$ c = x_{-m+1} = \ldots = x_0 < x_1 < \ldots < x_n = \ldots = x_{n+m-1} = d, \quad (2.1) $$

an equally-spaced knots sequence [23]. The functions

$$ B_{m,k;X} (x) = \frac{x - x_k}{x_{k+m-1} - x_k} B_{m-1,k;X} (x) + \frac{x_{k+m} - x}{x_{k+m} - x_{k+1}} B_{m-1,k+1;X} (x), $$

$$ k = -m + 1, \ldots, n - 1, \quad (2.2) $$

and

$$ B_{1,k;X} (x) = \begin{cases} 1, & x \in [x_k, x_{k+1}), \\ 0, & \text{otherwise}, \end{cases} \quad k = 0, \ldots, n - 1, \quad (2.3) $$

are called cardinal B-spline functions of order $m \geq 2$ for the knots sequence $X = \{x_i\}_{i=-m+1}^{n+m-1}$ and $\text{supp}[B_{m,k;X} (x)] = [x_k, x_{k+m}] \cap [c, d]$ and $x_k = k, k = 0, \ldots, n$ are interior B-spline functions, while the remaining $B_{m,k;X} (x), k = -m + 1, \ldots, -1$ and $k = n - m + 1, \ldots, n - 1$ are boundary B-spline functions [22] for the bounded interval [0, 1]. Since the boundary B-spline functions at 0 are symmetric reflections of those at $n$, it is sufficient to construct only the first half functions by simply replacing $x$ with $n - x$.

By considering the interval $[c, d] \equiv [0, 1]$, at any level $j \in \mathbb{Z}^+$, the discretization step is $2^{-j}$, and this generates $n = 2^j$ number of segments in [0, 1] with knots sequence [22]

$$ X^{(j)} = \{ x_{-m+1}^{(j)} = \ldots = x_0^{(j)} = 0, $$

$$ x_k^{(j)} = k/2^j, $$

$$ x_n^{(j)} = \ldots = x_0^{(j)} = 1, \quad k = 1, \ldots, n - 1. \quad (2.4) $$

Let $j_0$ be the level for which $2^{j_0} \geq m$; for each level $j \geq j_0$ the scaling functions of order $m$ can be defined as follows:

$$ \phi_{m,k}^{(j)} (x) = \begin{cases} B_{m,k;X^{(j)}} (2^{-j_0} x), & k = -m + 1, \ldots, -1, \\ B_{m,2^{-j_0} - m;X^{(j)}} (1 - 2^{-j_0} x), & k = 2^j - m + 1, \ldots, 2^j - 1, \\ B_{m,0;X^{(j)}} (2^{-j_0} x - 2^{-j_0} k), & k = 0, \ldots, 2^j - m. \end{cases} \quad (2.5) $$

The scaling functions $\phi_{m,k}^{(j)}$ occupy $m$ segments, therefore the condition $2^j \geq m$ must be satisfied in order to have at least one inner scaling function [23].

In the following, the scaling functions used in this article, for $m = 4$ (cubic B-spline scaling functions) and $j \geq j_0 = 2$ are reported [22]:

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Boundary scaling

\[
\phi_{4,-3}(x) = \begin{cases} 
-(4x - 1)^3, & 0 \leq x < 1/4, \\
0, & \text{otherwise}, 
\end{cases} 
\] (2.6)

\[
\phi_{4,-2}(x) = \begin{cases} 
12x - 72x^2 + 112x^3, & 0 \leq x < 1/4, \\
-1/4(4x - 2)^3, & 1/4 \leq x < 1/2, \\
0, & \text{otherwise}, 
\end{cases} 
\] (2.7)

\[
\phi_{4,-1}(x) = \begin{cases} 
24x^2 - 176/3x^3, & 0 \leq x < 1/4, \\
18x - 48x^2 + 112/3x^3 - 3/2, & 1/4 \leq x < 1/2, \\
-1/6(4x - 3)^3, & 1/2 \leq x < 3/4, \\
0, & \text{otherwise}, 
\end{cases} 
\] (2.8)

\[
\phi_{4,j}(x) = \phi_{4,j}(2j-2x), \quad j = 2, 3, \ldots 
\] (2.9)

and

\[
\phi_{4,1}(x) = \phi_{4,-1}(1 - x), \\
\phi_{4,2}(x) = \phi_{4,-2}(1 - x), \\
\phi_{4,3}(x) = \phi_{4,-3}(1 - x), \\
\phi_{4,2j-\ell-3}(x) = \phi_{4,\ell}(1 - x), \quad \ell = -3, -2, -1, \quad j = 2, 3, \ldots 
\] (2.10)

Inner scaling

\[
\phi_{4,0}(x) = \begin{cases} 
32/3x^3, & 0 \leq x < 1/4, \\
32x^2 - 32x^3 - 8x + 2/3, & 1/4 \leq x < 1/2, \\
40x - 64x^2 + 32x^3 - 22/3, & 1/2 \leq x < 3/4, \\
-1/6(4x - 4)^3, & 3/4 \leq x \leq 1, \\
0, & \text{otherwise}, 
\end{cases} 
\] (2.11)

and

\[
\phi_{4,\ell}(x) = \phi_{4,0}(2^{j-2}x - \ell), \quad \ell = 0, \ldots, 2^j - 3, \quad j = 2, 3, \ldots 
\] (2.12)

A. Function Approximation

A function \( f(x) \) defined over \([0,1]\) may be represented by the cubic B-spline scaling functions as

\[
f(x) = \sum_{k=-3}^{2^{M-1}} s_k \phi_{4,k}^{(M)} = S^T \Phi_M, 
\] (2.13)

where

\[
S = [s_{-3}, s_{-2}, \ldots, s_{2^{M-1}}]^T, 
\] (2.14)

\[
\Phi_M = [\phi_{4,-3}^{(M)}(x), \phi_{4,-2}^{(M)}(x), \ldots, \phi_{4,2^{M-1}}^{(M)}(x)]^T, 
\] (2.15)
with
\[ s_k = \int_0^1 f(x) \tilde{\phi}_k^{(M)}(x) dx, \quad k = -3, \ldots, 2^M - 1, \]
where \( \tilde{\phi}_k^{(M)}(x) \) are dual functions of \( \phi_k^{(M)}(x) \). These can be obtained by linear combinations of \( \phi_k^{(M)}(x) \), as follows. Let \( \tilde{\Phi}_M \) be the dual functions of \( \Phi_M \) given by
\[ \tilde{\Phi}_M = [\tilde{\phi}_4^{(M)}, \ldots, \tilde{\phi}_{2^M-1}^{(M)}]^T. \] (2.16)
Eqs. (2.15) and (2.16) give
\[ \int_0^1 \tilde{\Phi}_{M+1}(x) \Phi_{M+1}^T(x) dx = I, \] (2.17)
where \( I \) is \( (2^M + 3) \times (2^M + 3) \) identity matrix. Using (2.6)–(2.12) and (2.15) we have
\[ \int_0^1 \Phi_M(x) \Phi_M^T(x) dx = P_M = \]
\[
\begin{bmatrix}
  p_1 & p_2 & p_3 & p_4 \\
p_2 & p_5 & p_6 & p_7 & p_8 \\
p_3 & p_6 & p_9 & p_{10} & p_{11} & \alpha \\
p_4 & p_7 & p_{10} & \eta & \gamma & \beta & \cdots \\
p_8 & p_{11} & \gamma & \cdots & \cdots & \cdots \\
\alpha & \beta & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha & \beta & \gamma & \eta & \gamma & \beta & \alpha \\
\alpha & \beta & \gamma & \eta & p_{11} & p_8 & \alpha \\
\alpha & \beta & \gamma & \eta & p_{10} & p_7 & p_4 \\
\alpha & p_{11} & p_{10} & p_9 & p_6 & p_5 \\
p_8 & p_7 & p_6 & p_5 & p_2 \\
p_4 & p_3 & p_2 & p_1 
\end{bmatrix}
\] (2.18)
where \( P_M \) is a symmetric \( (2^M + 3) \times (2^M + 3) \) matrix and \( p_1 = 1/7, p_2 = 7/80, p_3 = 31/1680, p_4 = 1/840, p_5 = 31/140, p_6 = 5/32, p_7 = 29/840, p_8 = 1/3360, p_9 = 183/560, p_{10} = 283/1260, p_{11} = 239/10080, \alpha = 1/5040, \beta = 1/42, \gamma = 397/1680, \eta = 151/315. \) From (2.17) and (2.18) we get
\[ \tilde{\Phi}_M = (P_M)^{-1} \Phi_M. \]
Now a function \( u(x,t) \) of two independent variables defined for \( 0 \leq x \leq 1 \) and \( 0 \leq t \leq 1 \) may be expanded in terms of double cubic B-spline scaling functions as
\[ u(x,t) = \sum_{i=-3}^{2^M-1} \sum_{j=-3}^{2^M-1} u_{ij} \phi_{4,i}^{(M)}(t) \phi_{4,j}^{(M)}(x) = \Phi_M^T(t) U \Phi_M(x), \] (2.19)
where $U$ is a $(2^M + 3) \times (2^M + 3)$ matrix as

$$
\begin{bmatrix}
    u_{-3,-3} & \cdots & u_{-3,2M-1} \\
    \vdots & & \vdots \\
    u_{2M-1,-3} & \cdots & u_{2M-1,2M-1}
\end{bmatrix},
$$

and

$$
u_{i,j} = \int_0^1 \int_0^1 u(x,t) \tilde{\phi}^{(M)}_{i,j}(t) \tilde{\phi}^{(M)}_{i,j}(x) dt dx, \quad i, j = -3, -2, \ldots, 2^M - 1.
$$

B. The Operational Matrices of Derivative

The differentiation of vectors $\Phi_M$ in (2.15) can be expressed as

$$
\Phi_M' = D \Phi_M,
$$

where $D$ is $(2^M + 3) \times (2^M + 3)$ operational matrix of derivative for B-spline scaling functions. The matrix $D$ can be obtained by considering

$$
D = \int_0^1 \Phi_M'(x) \Phi_M^T(x) dx = \left( \int_0^1 \Phi_M'(x) \Phi_M^T(x) dx \right) (P_M)^{-1} = EP_M^{(-1)},
$$

where

$$
E = \int_0^1 \Phi_M'(x) \Phi_M^T(x) dx.
$$

In (2.22), $E$ is a $2^M \times 2^M$ matrix given by

$$
E = \begin{bmatrix}
    \int_0^1 \phi_i^{(M)}(x) \phi_j^{(M)}(x) dx & \cdots & \int_0^1 \phi_i^{(M)}(x) \phi_{2M-1}^{(M)}(x) dx \\
    \vdots & \ddots & \vdots \\
    \int_0^1 \phi_{2M-1}^{(M)}(x) \phi_i^{(M)}(x) dx & \cdots & \int_0^1 \phi_{2M-1}^{(M)}(x) \phi_{2M-1}^{(M)}(x) dx
\end{bmatrix}.
$$

By calculating the entries of matrix $E$ we get

$$
E = \begin{bmatrix}
    -d_1 & -d_2 & -d_3 & -d_4 \\
    d_2 & 0 & -d_5 & -d_6 & -d_7 \\
    d_3 & d_5 & 0 & -d_8 & -d_9 & -\lambda_1 \\
    d_4 & d_6 & d_8 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\
    d_7 & d_9 & \lambda_3 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -\lambda_3 & -\lambda_2 & -\lambda_1 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
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    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
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    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7 \\
    \lambda_1 & \lambda_2 & \lambda_3 & 0 & -d_8 & -d_9 & -d_7
\end{bmatrix}
$$

(2.24)
where $d_1 = 1/2, d_2 = 31/80, d_3 = 5/48, d_4 = 1/120, d_5 = 133/480, d_6 = 13/120, d_7 = 1/480, d_8 = 109/360, d_9 = 37/480, \lambda_1 = 1/720, \lambda_2 = 7/90, \lambda_3 = 49/144$.

### III. DESCRIPTION OF THE NEW NUMERICAL METHOD

Consider the Fokker—Planck equation (1.8). This equation can be rewritten as

$$\frac{\partial u}{\partial t} = -u \frac{\partial}{\partial x} A - A \frac{\partial u}{\partial x} + u \frac{\partial^2}{\partial x^2} B + 2 \frac{\partial u}{\partial x} \frac{\partial}{\partial u} B + B \frac{\partial^2 u}{\partial x^2}. \quad (3.1)$$

where $A = A(x, t, u)$ and $B = B(x, t, u)$. Also using the well-known chain rule we can write

$$\frac{\partial}{\partial x} A(x, t, u) = \frac{\partial A}{\partial x} + \frac{\partial A}{\partial u} \frac{\partial u}{\partial x}, \quad (3.2)$$

and

$$\frac{\partial^2}{\partial x^2} B = \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial x \partial u} \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial u^2} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial B}{\partial u} \frac{\partial u}{\partial x}. \quad (3.4)$$

Using Eqs. (2.19) and (2.20) we have:

$$\frac{\partial u}{\partial t} = \Phi_T^T(t) D^T U \Phi_M(x), \quad (3.5)$$

$$\frac{\partial u}{\partial x} = \Phi_T^T(t) D U \Phi_M(x), \quad (3.6)$$

$$\frac{\partial^2 u}{\partial x^2} = \Phi_T^T(t) D^2 \Phi_M(x). \quad (3.7)$$

Using Eqs. (3.2)–(3.7) in Eq. (3.1), we get:

$$\Phi_T^T(t) D^T U \Phi_M(x) = -u \left[ \frac{\partial A}{\partial x} + \frac{\partial A}{\partial u} \Phi_M^T(t) U D \Phi_M(x) \right] - A \Phi_T^T(t) U D \Phi_M(x) + u \left[ \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial x \partial u} \Phi_M^T(t) U D \Phi_M(x) \right] + \Phi_M^T(t) U D^2 \Phi_M(x) \cdot \frac{\partial B}{\partial u} + \frac{\partial^2 B}{\partial u \partial x} \cdot \Phi_M^T(t) U D \Phi_M(x) \right] + 2 \Phi_T^T(t) U D \Phi_M(x) \cdot \frac{\partial B}{\partial x} + \frac{\partial B}{\partial u} \Phi_M^T(t) U D \Phi_M(x) + B \Phi_M^T(t) U D^2 \Phi_M(x). \quad (3.8)$$

Also applying (2.19) in the initial condition (1.3) we get

$$\Phi_T^T(0) U \Phi_M(x) = f(x). \quad (3.9)$$

By collocating Eq. (3.8) in $(2^M + 3) \times (2^M + 2)$ points $(x_i, t_j), i = 1, 2, \ldots, 2^M + 3, j = 1, 2, \ldots, 2^M + 2$ on $[0, 1] \times [0, 1]$ and Eq. (3.9) in $2^M + 3$ points $x_i, i = 1, 2, \ldots, 2^M + 3$ on $[0, 1]$, we get an algebraic system of $(2^M + 3) \times (2^M + 2)$ equations and unknowns that can be solved for $u_{i,j}, i, j = 1, 2, \ldots, 2^M + 3$. So the unknown function $u(x, t)$ can be found.

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IV. NUMERICAL EXAMPLES

In this section, we give some computational results of numerical experiments with method based on preceding sections, to support our theoretical discussion. The nonlinear systems obtained by the new technique are solved by Newton method and the collocation points are considered with equal space on the interval. It is worth pointing out that we solve the problem on a bounded interval.

Example 1. Consider [2,4] equation (1.3) with:

\[ f(x) = x, \quad x \in [0, 1]. \]

Let in Eq. (1.2) we choose \( A(x) = -1 \), and \( B(x) = 1 \). The exact solution of this problem is \( u(x, t) = x + t \). Table I shows the absolute error of the method presented in the previous section for \( M = 3 \).

Example 2. Consider [2,4] equation (1.2) with \( A(x) = x, B(x) = x^2/2 \) and \( f(x) = x \). The exact solution of this problem is \( u(x, t) = x \exp(t) \). Table II shows the absolute error using the procedure proposed in the previous section for \( M = 3 \).

Example 3. Consider [2,4] the backward Kolmogorov equation (1.5) with drift and diffusion coefficients given respectively by:

\[ A(x, t) = -(x + 1), \]
\[ B(x, t) = x^2 \exp(t). \]

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<th>( x = 0.4 )</th>
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<td>1.2 \times 10^{-10}</td>
<td>1.7 \times 10^{-10}</td>
</tr>
<tr>
<td>0.005</td>
<td>1.6 \times 10^{-10}</td>
<td>2.5 \times 10^{-10}</td>
<td>3.9 \times 10^{-10}</td>
<td>5.8 \times 10^{-10}</td>
<td>8.2 \times 10^{-10}</td>
</tr>
<tr>
<td>0.01</td>
<td>3.1 \times 10^{-10}</td>
<td>4.8 \times 10^{-10}</td>
<td>7.4 \times 10^{-10}</td>
<td>1.1 \times 10^{-9}</td>
<td>1.6 \times 10^{-9}</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2 \times 10^{-9}</td>
<td>1.8 \times 10^{-9}</td>
<td>2.7 \times 10^{-9}</td>
<td>3.8 \times 10^{-9}</td>
<td>5.0 \times 10^{-9}</td>
</tr>
<tr>
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<td>3.1 \times 10^{-10}</td>
<td>4.8 \times 10^{-10}</td>
<td>7.4 \times 10^{-10}</td>
<td>1.1 \times 10^{-9}</td>
<td>1.6 \times 10^{-9}</td>
</tr>
<tr>
<td>0.15</td>
<td>2.0 \times 10^{-9}</td>
<td>2.6 \times 10^{-9}</td>
<td>3.1 \times 10^{-9}</td>
<td>2.9 \times 10^{-9}</td>
<td>1.3 \times 10^{-9}</td>
</tr>
<tr>
<td>1.00</td>
<td>1.3 \times 10^{-6}</td>
<td>2.0 \times 10^{-6}</td>
<td>3.0 \times 10^{-6}</td>
<td>4.4 \times 10^{-6}</td>
<td>6.6 \times 10^{-6}</td>
</tr>
</tbody>
</table>

Table III. Absolute values of errors for $u(x, t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x = 0.2$</th>
<th>$x = 0.4$</th>
<th>$x = 0.6$</th>
<th>$x = 0.8$</th>
<th>$x = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$3.1 \times 10^{-8}$</td>
<td>$3.6 \times 10^{-8}$</td>
<td>$4.0 \times 10^{-8}$</td>
<td>$4.5 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.005</td>
<td>$1.1 \times 10^{-7}$</td>
<td>$1.3 \times 10^{-7}$</td>
<td>$1.5 \times 10^{-7}$</td>
<td>$1.7 \times 10^{-7}$</td>
<td>$1.9 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$1.8 \times 10^{-7}$</td>
<td>$2.1 \times 10^{-7}$</td>
<td>$2.4 \times 10^{-7}$</td>
<td>$2.7 \times 10^{-7}$</td>
<td>$3.0 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.05</td>
<td>$2.0 \times 10^{-8}$</td>
<td>$2.3 \times 10^{-8}$</td>
<td>$2.7 \times 10^{-8}$</td>
<td>$3.0 \times 10^{-8}$</td>
<td>$3.3 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.10</td>
<td>$1.8 \times 10^{-7}$</td>
<td>$2.1 \times 10^{-7}$</td>
<td>$2.4 \times 10^{-7}$</td>
<td>$2.7 \times 10^{-7}$</td>
<td>$3.0 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.15</td>
<td>$1.9 \times 10^{-7}$</td>
<td>$2.2 \times 10^{-7}$</td>
<td>$2.6 \times 10^{-7}$</td>
<td>$2.9 \times 10^{-7}$</td>
<td>$3.2 \times 10^{-7}$</td>
</tr>
<tr>
<td>1.00</td>
<td>$7.4 \times 10^{-7}$</td>
<td>$8.7 \times 10^{-7}$</td>
<td>$9.9 \times 10^{-7}$</td>
<td>$1.1 \times 10^{-6}$</td>
<td>$1.2 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Let the initial condition in (1.3) be given by:

$$f(x) = x + 1, \quad x \in [0, 1].$$

The exact solution of this problem is $u(x, t) = (x + 1) \exp(t)$. Table III shows the absolute error using the technique presented in previous section for $M = 3$.

**Example 4.** Consider [2,4] the nonlinear Fokker-Planck equation (1.8) with:

$$A(x, t, u) = \frac{4}{x} - \frac{x}{3},$$

$$B(x, t, u) = u,$$

and

$$f(x) = x^2.$$

The exact solution of this problem is $u(x, t) = x^2 \exp(t)$. Table IV shows the absolute error using the method introduced in previous section for $M = 3$.

**Example 5.** Consider [2,4] the nonlinear Fokker-Planck equation (1.8) with:

$$A(x, t, u) = \frac{7}{2}u,$$

$$B(x, t, u) = xu,$$

and

$$f(x) = x.$$

Table IV. Absolute values of errors for $u(x, t)$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x = 0.2$</th>
<th>$x = 0.4$</th>
<th>$x = 0.6$</th>
<th>$x = 0.8$</th>
<th>$x = 1.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>$8.9 \times 10^{-10}$</td>
<td>$3.6 \times 10^{-9}$</td>
<td>$8.0 \times 10^{-9}$</td>
<td>$1.4 \times 10^{-8}$</td>
<td>$2.2 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.005</td>
<td>$3.8 \times 10^{-9}$</td>
<td>$1.5 \times 10^{-8}$</td>
<td>$3.4 \times 10^{-8}$</td>
<td>$6.0 \times 10^{-8}$</td>
<td>$9.4 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$6.0 \times 10^{-9}$</td>
<td>$2.4 \times 10^{-8}$</td>
<td>$5.4 \times 10^{-8}$</td>
<td>$9.6 \times 10^{-8}$</td>
<td>$1.5 \times 10^{-7}$</td>
</tr>
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<td>$6.4 \times 10^{-10}$</td>
<td>$2.6 \times 10^{-9}$</td>
<td>$5.8 \times 10^{-9}$</td>
<td>$1.0 \times 10^{-8}$</td>
<td>$1.6 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.10</td>
<td>$6.0 \times 10^{-9}$</td>
<td>$2.4 \times 10^{-8}$</td>
<td>$5.4 \times 10^{-8}$</td>
<td>$9.6 \times 10^{-8}$</td>
<td>$1.5 \times 10^{-7}$</td>
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<tr>
<td>0.15</td>
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<td>$5.8 \times 10^{-8}$</td>
<td>$1.0 \times 10^{-7}$</td>
<td>$1.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>1.00</td>
<td>$2.5 \times 10^{-8}$</td>
<td>$9.8 \times 10^{-8}$</td>
<td>$2.2 \times 10^{-7}$</td>
<td>$3.9 \times 10^{-7}$</td>
<td>$6.1 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
The exact solution of this problem is \( u(x, t) = \frac{x}{t+1} \). Table IV shows the absolute error using the scheme given in the previous section for \( M = 3 \).

Also note that we have taken the examples from the literature [2, 4].

V. CONCLUSION

The Fokker-Planck partial differential equation has wide applications in several areas in science and engineering. For example, this equation is an important tool in dynamics and is used extensively for modeling single particle in accelerators under the influence of noise. Thus, it is important to find a reliable numerical technique for solving this time-dependent partial differential equation. In this article, we presented a numerical scheme for solving the Fokker–Planck equation. The cubic B-spline scaling function together with boundary scaling function on interval \([0, 1]\) employed to solve the studied model. The obtained results showed that this approach can solve the problem effectively.

References

